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LETTER TO THE EDITOR

Hamiltonian path integrals in *n* dimensions

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Abstract. Using Hamiltonian path integrals we obtain the spurious potential term that arises in the action when a path integral is transformed to general n-dimensional coordinates.

It is widely recognised that when a non-linear point transformation is applied to a path integral the effective action generally picks up a spurious potential term of order $\hbar^2$ (DeWitt 1957). A concrete example of this phenomenon is seen in the transformation of the path integral for a free particle in cartesian coordinates to polar coordinates (Edwards and Gulyaev 1964, Peak and Inomata 1968, Lee 1981). The precise form of this term depends on how the path integral is formulated (Salomonson 1977).

Point transformations have always played a special role in physics. The difficulties attendant to their application to path integrals can be readily appreciated by noting that only recently has the hydrogen atom been solved within this formulation (Ho and Inomata 1982). In many instances canonical transformations, which are best developed with a Hamiltonian formulation, lead to deeper understanding of physical theories.

In general the derivation of the spurious potential term is based on a Lagrangian path integral (Gervais and Jevicki 1976, Mayes and Dowker 1972, Arthurs 1970a, b). Over the past decade, however, there has been much interest in Hamiltonian path integrals (a discussion is given in Schulman (1981)). In this letter we obtain an expression for this term in a general n-dimensional coordinate system using Hamiltonian path integrals and following a procedure advocated by Kapoor (1984). We also show that the result is identical to that obtained using the canonical procedure. The potential does not vanish even in flat space.

By Hamiltonian path integral we understand the form

$$\langle q''t''|q't'\rangle = \int \mathrm{d}p\langle q''t''|pt\rangle\langle pt|q't'\rangle$$

for the transition amplitude. Here $\langle q''t''|pt\rangle = \langle q''|\mathrm{e}^{-\mathrm{i}H(t''-t)}|p\rangle$, etc. For Lagrangian path integrals one uses $\int|qt\rangle\mathrm{d}q\langle qt|$ for the unit operator (Garrod 1966).

We begin with a free particle of unit mass in n dimensions whose Lagrangian takes the form

$$L = \tfrac{1}{2}g_{ij}\dot{q}^i\dot{q}^j \qquad i, j = 1, 2, \ldots, n \tag{1}$$

where the summation convention is implied. The g_{ij} are components of the metric tensor which we consider to be a function of the coordinates q^i. Dots stand for differentiation with respect to time t. The equations of motion are

$$\ddot{q}^i = -\Gamma^i_{jk}\dot{q}^j\dot{q}^k \tag{2}$$

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where the Γ^i_{jk} are the Christoffel symbols

$$\Gamma^i_{jk} = \tfrac{1}{2} g^{im}(g_{jm,k} + g_{km,j} - g_{jk,m})$$
$$g_{jm,k} = \partial g_{jm}/\partial q^k.$$

Equation (2) allows us to show that L is time independent so that the action S for a small time interval ε may be written

$$S[q(t+\varepsilon);\, q(t)] = \int_t^{t+\varepsilon} L\, \mathrm{d}t = \tfrac{1}{2}\varepsilon g_{ij}(t)\dot{q}^i(t)\dot{q}^j(t). \tag{3}$$

The momenta conjugate to q^i are

$$p_i = \partial L/\partial \dot{q}^i = g_{ij}\dot{q}^j. \tag{4}$$

Following DeWitt (1957) we will adopt the convention

$$\begin{aligned} q'^i &\equiv q^i(t) \qquad & p'_i &\equiv p_i(t) \\ q''^i &\equiv q^i(t+\varepsilon) \qquad & p''_i &\equiv p_i(t+\varepsilon) \end{aligned} \tag{5}$$

and similarly for functions of these variables.

In what follows it will be convenient to write (3) as a function of q'' and of p'. To this end we expand g'^{ij} as follows:

$$\begin{aligned} g'^{ij} &= g^{ij}(t) = g^{ij}(t+\varepsilon) - \varepsilon\dot{q}^k(t+\varepsilon) g^{ij}_{,k}(t+\varepsilon) + \mathrm{O}(\varepsilon^2) \\ &= g''^{ij} - \varepsilon p'_m g''^{km} g''^{ij}_{,k} + \mathrm{O}(\varepsilon^2). \end{aligned} \tag{6}$$

Similarly we have

$$\begin{aligned} q'^i &= q''^i - \varepsilon\dot{q}'^i - \tfrac{1}{2}\varepsilon^2\ddot{q}'^i \\ &= q''^i - \varepsilon g'^{ij} p'_j + \tfrac{1}{2}\varepsilon^2 \Gamma'^i_{mn} g'^{ml} g'^{nj} p'_l p'_j \\ &= q''^i - \varepsilon g''^{ij} p'_j + \varepsilon^2(\tfrac{1}{2}\Gamma''^i_{mn} g''^{ml} g''^{nj} + g''^{kl} g''^{ij}_{,k}) p'_l p'_j + \mathrm{O}(\varepsilon^3) \end{aligned} \tag{7}$$

where we have used (2) and (6). We may now write the action as

$$S[q'';\, q'] = \tfrac{1}{2}\varepsilon p'_i \dot{q}'^i = \tfrac{1}{2}\varepsilon p'_i [g''^{ij} - \varepsilon p'_m g''^{km} g''^{ij}_{,k} + \ldots] p'_j. \tag{8}$$

We are now in a position to write an expression for the transition amplitude. The transition amplitude

$$\langle q'', t'' | q', t' \rangle = \int \mathrm{d}p \langle q'', t'' | p, t \rangle \langle p, t | q', t' \rangle \tag{9}$$

may be evaluated by using (DeWitt 1957, Kapoor 1984) the following forms for the short-time propagators:

$$\begin{aligned} \langle q'', t'' | p, t'' - \varepsilon \rangle &= \frac{1}{g''^{1/4}} D^{1/2}_{++}\, \mathrm{e}^{\mathrm{i}S_{++}[q''t'';\, pt]} \\ \langle p, t' + \varepsilon | q', t' \rangle &= \frac{1}{g'^{1/4}} D^{1/2}_{--}\, \mathrm{e}^{\mathrm{i}S_{--}[p,t'+\varepsilon;\, q',t']} \end{aligned} \tag{10}$$

where $t = t'' - \varepsilon$, g is the determinant of the metric tensor and $S_{++}(S_{--})$ is the generator of the canonical transformation taking $p(t)$ to $q(t'')(q(t')$ to $p(t))$. The D are the van Vleck determinants

$$D_{++} = \det\left|\frac{\partial^2 S_{++}}{\partial q''\, \partial p}\right| \qquad D_{--} = \det\left|\frac{\partial^2 S_{--}}{\partial q'\, \partial p}\right|$$

where we have denoted the coordinates and momenta at time t by q^i and p_i. We also have

$$S_{++}[q''; p] = pq + S[q''; q] \tag{11}$$

where $S[q'', q]$ is the action (2).

With the aid of (7) we may write

$$S_{++}[q''; p] = p_i q''^i - \tfrac{1}{2} g''^{ij} p_i p_j + \tfrac{1}{2}\varepsilon^2 (\Gamma''^{i}_{mn} g''^{ml} g''^{nj} + g''^{kl} g''^{ij}_{,k}) p_i p_j p_l + \mathrm{O}(\varepsilon^3). \tag{12}$$

The van Vleck determinant is then

$$\begin{aligned} D_{++} &= \det\left|\frac{\partial^2 S_{++}}{\partial q''^j \partial p^i}\right| \\ &= \det|\delta^i_j - \varepsilon g''^{ik}_{,j} p_k - \tfrac{1}{2}\varepsilon^2 W''^{ilk}_j p_l p_k + \ldots| \end{aligned} \tag{13}$$

where

$$W''^{ilk}_j = (\Gamma''^{i}_{mn} g''^{ml} g''^{nk})_{,j} + (\text{cyclic combinations of } i, l, k)$$

making use of the expansion

$$\det|1 + B| = 1 + \mathrm{Tr}\, B + \tfrac{1}{2}(\mathrm{Tr}\, B)^2 - \tfrac{1}{2}\mathrm{Tr}\, B^2 + \ldots$$

we find

$$D_{++} = 1 - \varepsilon g''^{ij}_{,i} p_j - \tfrac{1}{2}\varepsilon^2 \Gamma''^{kij}_k p_i p_j + \ldots$$

where

$$\Gamma''^{kij}_k = -g''^{kl}_{,k} g''^{ij}_{,l} - g''^{jl}_{,k} g''^{ki}_{,l} - g''^{kj}_{,k} g''^{li}_{,l} + \Gamma''^{k}_{mn,k} g''^{mi} g''^{nj} + \Gamma''^{i}_{mn,k} g''^{km} g''^{nj} + \Gamma''^{j}_{mn,k} g''^{km} g''^{ni}. \tag{14}$$

We will denote the integrand of (9) by K and employ the expressions (10) for them. To obtain the equation satisfied by the transition amplitude let us consider the expression

$$\left[-\frac{1}{2}\frac{1}{g''^{1/2}}\frac{\partial}{\partial q''^m}\left(g''^{1/2} g''^{mn}\frac{\partial}{\partial q''^n}\right) - \mathrm{i}\frac{\partial}{\partial \varepsilon}\right] K$$

where

$$\Delta = \frac{1}{g^{1/2}}\frac{\partial}{\partial q^m}\left(g^{1/2} g^{mn}\frac{\partial}{\partial q^n}\right)$$

is just the Laplace operator in n dimensions (Marinov 1980). Some simplification in the above may be effected if we use the Hamilton–Jacobi equation (DeWitt 1957)

$$\frac{\partial S_{++}}{\partial t} + H\left(q, \frac{\partial S_{++}}{\partial q}, t\right) = 0 \tag{15}$$

H being the Hamiltonian. In the limit $\varepsilon \to 0$ we obtain

$$\left(H - \mathrm{i}\frac{\partial}{\partial t}\right) k = \tfrac{1}{4}(g^{ln}_{,l}\Gamma^k_{nk} + \tfrac{1}{2} g^{ln}\Gamma^k_{lk}\Gamma^j_{nj} + g^{ln}\Gamma^i_{ni,l}) K. \tag{16}$$

It is not difficult to see that the same equation is satisfied by the transition amplitude. This is our generalisation of Kapoor's result. The effect of a scalar potential $v(q)$ is to include a term $-vK$ on the right-hand side of (16).

The term on the right-hand side of (16) is the extra potential term. In general it is non-zero even in flat space. To see this we rewrite it as

$$\tfrac{1}{4}[-R-\tfrac{1}{2}g^{mn}(\Gamma^{l}_{nl}\Gamma^{j}_{mj}-2\Gamma^{l}_{nm,l}+\Gamma^{l}_{mj}\Gamma^{j}_{nl})]K \tag{17}$$

where R is the scalar curvature

$$R=g^{kl}(\Gamma^{i}_{kl,i}-\Gamma^{i}_{ki,l}+\Gamma^{i}_{jl}\Gamma^{j}_{kl}-\Gamma^{i}_{jl}\Gamma^{j}_{ki}).$$

The extra potential term is independent of the external potential: it is a property of the coordinate system used. For polar coordinates it is just $-(8r^2)^{-1}K$. For spherical coordinates it is $-(8r^2)^{-1}(1+\operatorname{cosec}^2\theta)K$.

Finally we derive (16) using the canonical operator method. For the free particle case we have (Marinov 1980)

$$\hat{q}^i,\hat{p}^i=-\mathrm{i}\left(\frac{\partial}{\partial q^i}+\frac{1}{2}\frac{\partial}{\partial q^i}\log g^{1/2}\right) \tag{18}$$

with the commutation relations

$$[\hat{q}^i,\hat{p}_j]=\mathrm{i}\delta^i_j. \tag{19}$$

The Hamiltonian operator $\hat{H}$ is given by

$$\hat{H}=\tfrac{1}{2}\hat{p}_i g^{ij}\hat{p}_j.$$

Expanding $\hat{H}$ we find

$$\hat{H}=-\tfrac{1}{2}\Delta+\tfrac{1}{4}(g^{ij}_{,i}\Gamma^{l}_{jl}+\tfrac{1}{2}g^{ij}\Gamma^{l}_{jl}\Gamma^{k}_{ik}+g^{ij}\Gamma^{l}_{jl,i}). \tag{20}$$

The second term arises from operator ordering and can be readily shown to be identical to the right-hand side of (16). Clearly this potential is of order $\hbar^2$.

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